

VOLUME CONJECTURE FOR $SU(n)$ -INVARIANTS

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ABSTRACT. This paper discuss an intrinsic relation among congruent relations [1], cyclotomic expansion and Volume Conjecture for $SU(n)$ invariants. Motivated by the congruent relations for $SU(n)$ invariants obtained in our previous work [1], we study certain limits of the $SU(n)$ invariants at various roots of unit. First, we prove a new symmetry property for the $SU(n)$ invariants by using a symmetry of colored HOMFLYPT invariants. Then we propose some conjectural formulas including the cyclotomic expansion conjecture and volume conjecture for $SU(n)$ invariants (specialization of colored HOMFLYPT invariants). We also give the proofs of these conjectural formulas for the case of figure-eight knot.

1. INTRODUCTION

In our previous work joint with P. Peng [1], we introduced the $SU(n)$ quantum invariant for a link \mathcal{L} as follow:

$$\begin{aligned}
 (1.1) \quad J_N^{SU(n)}(\mathcal{L}; q) &= \left(\frac{q^{-2lk(\mathcal{L})\kappa(N)} t^{-2lk(\mathcal{L})N} W_{(N)(N), \dots, (N)}(\mathcal{L}; q, t)}{s_{(N)}(q, t)} \right) \Big|_{t=q^n} \\
 &= \frac{q^{-2lk(\mathcal{L})N(N-1)} q^{-2nlk(\mathcal{L})N} W_{(N)(N), \dots, (N)}(\mathcal{L}; q, q^n)}{s_{(N)}(q, q^n)} \\
 &= \frac{q^{-2lk(\mathcal{L})(N(N-1)+nN)} W_{(N)(N), \dots, (N)}(\mathcal{L}, q, q^n)}{s_{(N)}(q, q^n)}
 \end{aligned}$$

where $W_{(N)(N), \dots, (N)}(\mathcal{L}, q, t)$ is the colored HOMFLYPT invariants of \mathcal{L} , see Section 2.1 for the definitions. In particular, when $n = 2$, $J_N^{SU(2)}(\mathcal{L}; q) = J_N(\mathcal{L}; q)$ is the classical (reduced) colored Jones polynomial with a suitable variable changes, see Section 7 in [1] for detail. In this paper, by using one of the symmetries of the colored HOMFLYPT invariants obtained in [1], we prove the following symmetry of the $SU(n)$ invariant about the rank n .

Theorem 1.1. *For a knot \mathcal{K} and two integers $n \geq m \geq 2$, we have*

$$(1.2) \quad J_N^{SU(n)}(\mathcal{K}; q) \equiv J_N^{SU(n-m)}(\mathcal{K}; q) \pmod{[m]}.$$

Remark 1.2. As in [1], we use the notation $[m] = q^m - q^{-m}$ throughout this paper, and $A \equiv B \pmod{[m]}$ denotes $\frac{A-B}{C} \in \mathbb{Z}[q, q^{-1}]$.

Formula (1.2) can be viewed as a new congruent relation with respect to the rank n . In [1], we have proposed the following congruent relation for $J_N^{SU(n)}(\mathcal{K}; q)$ which reveals some symmetries with respect to the color N :

$$(1.3) \quad J_N^{SU(n)}(\mathcal{K}; q) - J_k^{SU(n)}(\mathcal{K}; q) \equiv 0 \pmod{[N-k][N+k+n]},$$

where $N \geq k \geq 0$. In fact, the congruent relation (1.3) is an easy consequence of the following more general conjecture.

Conjecture 1.3 (cyclotomic expansions for $SU(n)$ invariants). *For any knot \mathcal{K} , there exist $H_k^{(n)}(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$, independent of N ($N \geq 0$). Such that*

$$(1.4) \quad J_N^{SU(n)}(\mathcal{K}; q) = \sum_{k=0}^N C_{N+1,k}^{(n)} H_k^{(n)}(\mathcal{K}),$$

where $C_{N+1,k}^{(n)} = [N - (k-1)][N - (k-2)] \cdots [N-1][N][N+n][N+n+1] \cdots [N+n+(k-1)]$, for $k = 1, \dots, N$, and $C_{N+1,0}^{(n)} = 1$. In particular, $J_0^{SU(n)}(\mathcal{K}; q) = H_0^{(n)}(\mathcal{K}) = 1$.

Conjecture 1.3 is a generalization of the cyclotomic expansion for colored Jones polynomials due to K. Habiro [3]. By some direct calculations, we find that Conjecture 1.3 holds for figure-eight knot 4_1 and trefoil knot 3_1 . See examples 2.5 and 2.6 in Section 2.

Next, we study the limit behaviors of $SU(n)$ invariant $J_N^{SU(n)}(\mathcal{K}; q)$ at various roots of unit. For convenience, we introduce the notation $\xi_{N,a}(s) = \exp(\frac{s\pi\sqrt{-1}}{N+a})$, where $a, s \in \mathbb{Z}$. Then for a fixed $n \geq 2$, we have

Conjecture 1.4. (i) *If $a \in \mathbb{Z} \setminus \{1, 2, \dots, n-1\}$, then for any knot \mathcal{K} :*

$$(1.5) \quad 2\pi s \lim_{N \rightarrow \infty} \frac{\log J_N^{SU(n)}(\mathcal{K}; \xi_{N,a}(s))}{N+1} = 0.$$

(ii) *If $a \in \{1, 2, \dots, n-1\}$, then*

$$(1.6) \quad 2\pi s \lim_{N \rightarrow \infty} \frac{\log J_N^{SU(n)}(\mathcal{K}; \xi_{N,a}(s))}{N+1} = \text{Vol}(S^3/\mathcal{K}) + \sqrt{-1}CS(S^3/\mathcal{K})$$

for any hyperbolic knot \mathcal{K} .

Conjecture 1.4 is a parallel generalization of the complex volume conjecture for colored Jones polynomial [9][6][7]. In fact, when $n = 2$, part (i) of the Conjecture 1.4 is an easy consequence of the congruent relation obtained in [1], see Section 2 for a proof. Moreover, for general $n \geq 2$, the congruent relation (1.3) leads to part (i) of the Conjecture 1.4 for $SU(n)$ invariant $J_N^{SU(n)}(\mathcal{K})$ similarly. As for part (ii) of Conjecture 1.4, we believe it also holds for hyperbolic links although we have only checked the case of knots. In the paper [11], K. Kawagoe first proposed a special case of the part (ii) of Conjecture 1.4 for $a = n-1$ and $s = 1$.

In Section 3, we will prove that

Theorem 1.5. *The Conjecture 1.4 holds for the figure-eight knot 4_1 .*

We note that in the $SU(n)$ invariants $J_N^{SU(n)}$, there are two variables N, n . So it is natural to consider the double limits $N, n \rightarrow \infty$. In [11], K. Kawagoe has studied the double limit of

$$(1.7) \quad 2\pi \lim_{N \rightarrow \infty} \frac{\log J_N^{SU(n)}(\mathcal{K}; \xi_{N,n-1}(1))}{N+1}$$

when $\frac{n-1}{N+n-1}$ keeps a different ratio. In this paper, we find certain double limit of the $SU(n)$ invariants $J_N^{SU(n)}$ will also converge to zero or to the volume of the hyperbolic knot complement. More precisely, we propose

Conjecture 1.6. Fix an integer $n \geq 2$,

(i) If $a \in \mathbb{Z} \setminus \{1, 2, \dots, n-1\}$, then for any knot \mathcal{K} :

$$(1.8) \quad 2\pi s \lim_{N \rightarrow \infty} \frac{\log J_N^{SU(N+a+n)}(\mathcal{K}; \xi_{N,a}(s))}{N+1} = 0.$$

(ii) If $a \in \{1, 2, \dots, n-1\}$, then

$$(1.9) \quad 2\pi s \lim_{N \rightarrow \infty} \frac{\log J_N^{SU(N+a+n)}(\mathcal{K}; \xi_{N,a}(s))}{N+1} = \text{Vol}(S^3/\mathcal{K}) + \sqrt{-1}CS(S^3/\mathcal{K})$$

for any hyperbolic knot \mathcal{K} .

As an application of Theorem 1.1, it is direct to prove

Theorem 1.7. Conjecture 1.6 is equivalent to Conjecture 1.4.

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2. SYMMETRY, CONGRUENT RELATIONS AND CYCLOTOMIC EXPANSIONS

2.1. Definitions. In this subsection, we first review the definition of the colored HOMFLYPT invariant of a link \mathcal{L} with L components $\mathcal{K}_1, \dots, \mathcal{K}_L$ (See Section 2 of [1] for detail).

Let $(\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0)$ be a Conway triple. The (unreduced framed) HOMFLYPT polynomial $\mathcal{H}(\mathcal{L}; q, t)$ of \mathcal{L} is determined by the following skein relation:

$$t\mathcal{H}(\mathcal{L}_+; q, t) - t^{-1}\mathcal{H}(\mathcal{L}_-; q, t) = (q - q^{-1})\mathcal{H}(\mathcal{L}_0; q, t)$$

with $\mathcal{H}(U; q, t) = \frac{t-t^{-1}}{q-q^{-1}}$, where U denotes an unknot.

The colored HOMFLYPT invariant can be defined by satellite knot. A satellite of a knot \mathcal{K} is $\mathcal{K} \star Q$ which means the knot \mathcal{K} decorated by a diagram Q in the annulus. (see Figure 1 for a framed trefoil \mathcal{K} decorated by skein element Q).

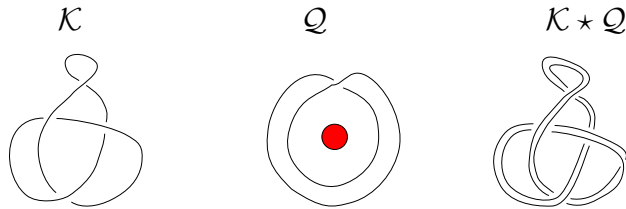


FIGURE 1.

There is a well-known set of idempotents, E_λ , for a partition λ of a positive integer n . The closure of E_λ , denoted by Q_λ forms a basis of the skein of annulus \mathcal{C} [8]. Given

L partitions $\lambda^1, \dots, \lambda^L$, let $\vec{\lambda} = (\lambda^1, \dots, \lambda^L)$, the colored HOMFLYPT invariant of \mathcal{L} (with color $\vec{\lambda}$) is defined as

$$W_{\vec{\lambda}}(\mathcal{L}; q, t) = q^{-\sum_{\alpha=1}^L \kappa_{\lambda^\alpha} w(\mathcal{K}_\alpha)} t^{-\sum_{\alpha=1}^L |\lambda^\alpha| w(\mathcal{K}_\alpha)} \mathcal{H}(\mathcal{L} \star \otimes_{\alpha=1}^L Q_{\lambda^\alpha}; q, t).$$

where $w(\mathcal{K}_\alpha)$ is the writhe number of knot $\mathcal{K}_{\lambda^\alpha}$, and $\kappa_\alpha = \sum_i \lambda_i^\alpha (\lambda_i^\alpha - 2i + 1)$, $|\lambda^\alpha| = \sum_i \lambda_i^\alpha$.

As to the unknot U , its colored HOMFLYPT invariant (with color λ) is given by

$$W_\lambda(U; q, t) = S_\lambda(q, t) = \sum_{\mu} \frac{\chi_\lambda(\mu)}{z_\mu} \prod_{j=1} \frac{t^{\mu_j} - t^{-\mu_j}}{q^{\mu_j} - q^{-\mu_j}}.$$

where $z_\mu = \prod_j \mu_j! |Aut(\mu)|$ and $\chi_\lambda(\mu)$ is the character symmetric group.

2.2. Symmetry. In [1], we proved several symmetries for colored HOMFLYPT invariants (see Theorem 1.2 in [1]), and one of them is

$$(2.1) \quad W_{\vec{\lambda}}(\mathcal{L}; q, -t) = (-1)^{\|\vec{\lambda}\|} W_{\vec{\lambda}}(\mathcal{L}; q, t).$$

for a given link \mathcal{L} with L components, and $\vec{\lambda} = (\lambda^1, \dots, \lambda^L) \in \mathcal{P}^L$,

As to a knot \mathcal{K} , we define the reduced colored HOMFLYPT invariant with symmetric representation corresponding to a partition (N) as follow

$$(2.2) \quad P_N(\mathcal{K}; q, t) = \frac{W_{(N)}(\mathcal{K}; q, t)}{S_{(N)}(q, t)}.$$

So by (2.1), we have

$$(2.3) \quad P_N(\mathcal{K}; q, t) = P_N(\mathcal{K}; q, -t).$$

Then, by the definition (1.1)

$$(2.4) \quad J_N^{SU(n)}(\mathcal{K}; q) = P_N(\mathcal{K}; q, q^n) = P_N(\mathcal{K}; q, -q^n).$$

Now we let $q = e^{\frac{s\pi\sqrt{-1}}{m}}$, where $s \in \mathbb{Z}$. If s odd, then

$$(2.5) \quad \begin{aligned} J_N^{SU(n)}(\mathcal{K}; e^{\frac{s\pi\sqrt{-1}}{m}}) &= P_N(\mathcal{K}; e^{\frac{s\pi\sqrt{-1}}{m}}, e^{\frac{n s \pi \sqrt{-1}}{m}}) \\ &= P_N(\mathcal{K}; e^{\frac{s\pi\sqrt{-1}}{m}}, -e^{\frac{(n-m)s\pi\sqrt{-1}}{m}}) \\ &= P_N(\mathcal{K}; e^{\frac{s\pi\sqrt{-1}}{m}}, e^{\frac{(n-m)s\pi\sqrt{-1}}{m}}) \\ &= J_N^{SU(n-m)}(\mathcal{K}; e^{\frac{s\pi\sqrt{-1}}{m}}), \end{aligned}$$

and if s even, it is obvious

$$(2.6) \quad J_N^{SU(n)}(\mathcal{K}; e^{\frac{s\pi\sqrt{-1}}{m}}) = J_N^{SU(n-m)}(\mathcal{K}; e^{\frac{s\pi\sqrt{-1}}{m}}).$$

In conclusion, we obtain

Theorem 2.1 (=Theorem 1.1). *For a knot \mathcal{K} , $n \geq m \geq 2$, we have*

$$(2.7) \quad J_N^{SU(n)}(\mathcal{K}; q) \equiv J_N^{SU(n-m)}(\mathcal{K}; q) \pmod{[m]}.$$

2.3. Congruent relations and cyclotomic expansions. At first, we review some results obtained in our previous joint work with P. Peng [1]. As to the colored Jones polynomial, we have proved the following congruent relations for $J_N(\mathcal{K}; q)$. See Theorem 7.3 in [1]:

Theorem 2.2. *For any knot \mathcal{K} , and $N \geq k \geq 0$, we have*

$$(2.8) \quad J_N(\mathcal{K}; q) - J_k(\mathcal{K}; q) \equiv 0 \pmod{[N-k][N+k+2]}.$$

In fact, it is a consequence of the cyclotomic expansion for the colored Jones polynomial due to K. Habiro [3]. For $N \geq 1$, we define

$$(2.9) \quad C_{N,k} = \prod_{j=1}^k (q^{2N} + q^{-2N} - q^{2j} - q^{-2j}).$$

In particular, $C_{N+1,0} = 1$, $C_{N+1,N+1} = 0$, and

$$\begin{aligned} C_{N+1,1} &= [N+2][N], \\ C_{N+1,2} &= [N+3][N+2][N][N-1], \\ &\dots \\ C_{N+1,k} &= [N+k+1][N+k] \cdots [N+2][N][N-1] \cdots [N-k+1] \\ &\dots \\ C_{N+1,N} &= [2N+1][2N] \cdots [N+2][N][N-1] \cdots [1]. \end{aligned}$$

In our notation, Habiro's cyclotomic expansion of colored Jones polynomial states:

Theorem 2.3 (K. Habiro [3]). *For any knot \mathcal{K} , there exist $H_k(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$, independent of N ($N \geq 0$). Such that*

$$(2.10) \quad J_N(\mathcal{K}; q) = \sum_{k=0}^N C_{N+1,k} H_k(\mathcal{K}).$$

In particular, $J_0(\mathcal{K}; q) = H_0(\mathcal{K}) = 1$.

In the next section, we will show that, by using Theorem 2.2, we can prove part (i) of Conjecture 1.4 for $n = 2$ case.

Now we consider the general $SU(n)$ invariants $J_N^{SU(n)}(\mathcal{K}; q)$. After some numerical computations, it is natural to propose the following conjecture which is the generalization of cyclotomic expansion (2.10) for colored Jones polynomial.

Conjecture 2.4 (=Conjecture 1.3). *For any knot \mathcal{K} , there exist $H_k^{(n)}(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$, independent of N ($N \geq 0$). Such that*

$$(2.11) \quad J_N^{SU(n)}(\mathcal{K}; q) = \sum_{k=0}^N C_{N+1,k}^{(n)} H_k^{(n)}(\mathcal{K}),$$

where $C_{N+1,k}^{(n)} = [N-(k-1)][N-(k-2)] \cdots [N-1][N][N+1] \cdots [N+n+(k-1)]$, for $k = 1, \dots, N$, and $C_{N+1,0}^{(n)} = 1$. In particular, $J_0^{SU(n)}(\mathcal{K}; q) = H_0^{(n)}(\mathcal{K}) = 1$.

Example 2.5. For the figure-eight knot 4_1 , by using formula (4) in [4], we have

$$(2.12) \quad H_k^{(n)}(4_1) = \frac{[n-2+k]!}{[k]![n-2]!} \in \mathbb{Z}[q, q^{-1}].$$

Example 2.6. For the trefoil knot 3_1 , by using formula (3.61) in [2] with $t = -1$, we have

$$(2.13) \quad H_k^{(n)}(3_1) = (-1)^k q^{k(2n+k-1)} \frac{[n-2+k]!}{[k]![n-2]!} \in \mathbb{Z}[q, q^{-1}].$$

See Appendix 4.2 for more examples of cyclotomic expansions of $SU(n)$ invariants.

By some direct calculations, we have

$$(2.14) \quad \begin{aligned} [N+n][N] - [k+n][k] &= [N-k][N+k+n], \\ [N+n+1][N-1] - [k+n+1][k-1] &= [N-k][N+k+n], \\ &\dots, \\ [N+k+n-1][N-k+1] - [2k+n-1][1] &= [N-k][N+k+n], \end{aligned}$$

for $N > k \geq 0$.

Therefore, if we assume Conjecture 2.4 holds, by similar method in the proof of Theorem 7.3 in [1], we obtain, for any knot \mathcal{K} and $N \geq k \geq 0$,

$$(2.15) \quad J_N^{SU(n)}(\mathcal{K}; q) - J_k^{SU(n)}(\mathcal{K}; q) \equiv 0 \pmod{[N-k][N+k+n]}.$$

which is the congruent relation for $SU(n)$ invariants obtained in [1]. See Conjecture 7.10 in [1].

Remark 2.7. After the completion of this paper, we note that in a recent paper by S. Nawata and A. Oblomkov [12], they propose a conjectural cyclotomic expansion formula for colored HOMFLYPT invariants with symmetric representation (See Conjecture 2.3 in [12]). Let $a = q^n$ in their conjecture, we obtain

Conjecture 2.8. For any knot \mathcal{K} , there exist $\tilde{H}_k(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$, independent of N with $N \geq 0$, such that

$$(2.16) \quad J_N^{SU(n)}(\mathcal{K}; q) = \tilde{H}_0(\mathcal{K}) + \tilde{C}_{N+1,1}^{(n)} \tilde{H}_1(\mathcal{K}) + \tilde{C}_{N+1,2}^{(n)} \tilde{H}_2(\mathcal{K}) + \dots + \tilde{C}_{N+1,N}^{(n)} \tilde{H}_N(\mathcal{K}),$$

where $\tilde{C}_{N+1,k}^{(n)} = \frac{1}{[k]!} \prod_{j=0}^{k-1} [N-j] \prod_{j=0}^{k-1} [N+n+j]$ for $k = 1, \dots, N$. In particular, $J_0^{SU(n)}(\mathcal{K}; q) = \tilde{H}_0(\mathcal{K}) = 1$.

It is clear that Conjecture 2.8 is a weak version of Conjecture 2.4.

3. CONGRUENT RELATIONS AND LIMITS OF $SU(n)$ INVARIANTS

In this section, we will show that the congruent relations is closely related to part (i) of Conjecture 1.4.

Before discussing the limit behaviors of the general $SU(n)$ invariants, we first review the classical volume conjecture [9] for colored Jones polynomials [6]. In our notations, it can be formulated as

Conjecture 3.1 (Complex volume conjecture). *Let \mathcal{L} be a hyperbolic link in S^3 , then*

$$(3.1) \quad 2\pi \lim_{N \rightarrow \infty} \frac{J_N(\mathcal{L}; \exp(\frac{\pi\sqrt{-1}}{N+1}))}{N+1} = \text{Vol}(S^3/\mathcal{L}) + CS(S^3/\mathcal{L}).$$

Many people have made a lot of efforts to prove the Conjecture 3.1, see [5] for a nice review. In the study of this conjecture, it is natural to think why we should take the unit root $q = \frac{\pi\sqrt{-1}}{N+1}$ in $J_N(\mathcal{L}; q)$, here the label N means the $N+1$ -dimensional irreducible representations of $U_q(sl_2)$. Namely, a natural question is what will happen if we take the other roots of unity in the above limit? In this section, we answer this question partially by the following theorem.

Theorem 3.2. *Given a knot \mathcal{K} in S^3 , let a and s be two fixed integer and $a \neq 1$. If we take $q = \xi_{N,a}(s) = \exp \frac{s\pi\sqrt{-1}}{N+a}$, then we have*

$$(3.2) \quad 2\pi s \lim_{N \rightarrow \infty} \frac{J_N(\mathcal{K}; \xi_{N,a}(s))}{N+1} = 0$$

Proof. In fact, it is an easy consequence of Theorem 2.2. For $a = 2, 3, 4, \dots$, by Theorem 2.2, we have

$$(3.3) \quad J_N(\mathcal{K}; \exp \frac{s\pi\sqrt{-1}}{N+a}) = J_{a-2}(\mathcal{K}; \exp \frac{s\pi\sqrt{-1}}{N+a})$$

For a fixed a , $J_{a-2}(\mathcal{K}, q)$ is a fixed polynomial of q , it is clear that the value of $J_{a-2}(\mathcal{K}; \exp \frac{s\pi\sqrt{-1}}{N+a})$ is bounded, i.e. there exists a constant C independent of the N such that

$$(3.4) \quad |J_{a-2}(\mathcal{K}; \exp \frac{s\pi\sqrt{-1}}{N+a})| < C.$$

So we have

$$(3.5) \quad \lim_{N \rightarrow \infty} \frac{J_N(\mathcal{K}; \exp \frac{s\pi\sqrt{-1}}{N+a})}{N+1} = \lim_{N \rightarrow \infty} \frac{J_{a-2}(\mathcal{K}; \exp \frac{s\pi\sqrt{-1}}{N+a})}{N+1} = 0$$

For $a = 0, -1, -2, -3, \dots$, then $-a = 0, 1, 2, 3, \dots$ Theorem 2.2 also gives

$$(3.6) \quad J_N(\mathcal{K}; \exp \frac{s\pi\sqrt{-1}}{N - (-a)}) = J_{-a}(\mathcal{K}; \exp \frac{s\pi\sqrt{-1}}{N+a}).$$

Similarly, we obtain

$$(3.7) \quad \lim_{N \rightarrow \infty} \frac{J_N(\mathcal{K}; \exp \frac{s\pi\sqrt{-1}}{N+a})}{N+1} = \lim_{N \rightarrow \infty} \frac{J_{-a}(\mathcal{K}; \exp \frac{s\pi\sqrt{-1}}{N+a})}{N+1} = 0$$

□

So we finish the proof of part (i) of Conjecture 1.4 for $n = 2$.

We note that when $a = 1$, the above limit is the complex volume by Conjecture 3.1. So in conclusion, the limit is equal to complex volume for $a = 1$ or is equal to zero if a takes the other integers. Why only $a = 1$ gives the nonzero limit? We observe that the mod term in formula (2.8) is given by $[N][N+2]$ if $k = 0$, and $N+1$ is the middle integer between N and $N+2$.

Now it is natural to consider the case of $SU(n)$ invariant $J_N^{SU(n)}(\mathcal{K})$. We observe that the mod term in the congruent relation formula (2.15) is now given by $[N][N+n]$ when $k=0$. So the middle integers between N and $N+n$ contain $N+1, N+2, \dots, N+n-1$. Therefore, as the application of congruent relation formula (2.15), similarly we have, if $a \in \mathbb{Z} \setminus \{1, 2, \dots, n-1\}$, then

$$(3.8) \quad 2\pi s \lim_{N \rightarrow \infty} \frac{J_N^{SU(n)}(\mathcal{K}; \xi_{N,a}(s))}{N+1} = 0.$$

Furthermore, as for $a \in \{1, 2, \dots, n-1\}$, we believe that the above limit is also convergent to the complex volume. So we propose the following conjecture:

Conjecture 3.3 (=Conjecture 1.4). *(i) If $a \in \mathbb{Z} \setminus \{1, 2, \dots, n-1\}$, then for any knot \mathcal{K} :*

$$(3.9) \quad 2\pi s \lim_{N \rightarrow \infty} \frac{\log J_N^{SU(n)}(\mathcal{K}; \xi_{N,a}(s))}{N+1} = 0.$$

(ii) If $a \in \{1, 2, \dots, n-1\}$, then

$$(3.10) \quad 2\pi s \lim_{N \rightarrow \infty} \frac{\log J_N^{SU(n)}(\mathcal{K}; \xi_{N,a}(s))}{N+1} = \text{Vol}(S^3/\mathcal{K}) + \sqrt{-1}CS(S^3/\mathcal{K})$$

for any hyperbolic knot \mathcal{K} .

In the following, we will prove

Theorem 3.4 (=Theorem 1.5). *The Conjecture 3.3 holds for the figure-eight knot 4_1 .*

The whole proof of the Theorem 3.4 is long, so we divide the Theorem 3.4 into the following two propositions and lemmas.

Lemma 3.5. *For $a=1$ or $n-1$, we have the following*

$$(3.11) \quad \lim_{N \rightarrow \infty} \frac{\log J_{N-1}^{SU(n)}(4_1; \xi_{N-1,a}(s))}{N} = \frac{2}{\pi s} \int_0^{\frac{5\pi}{6}} \log 2 \sin(x) dx.$$

Proof. By formula (4) in [4], we have

$$(3.12) \quad J_{N-1}^{SU(n)}(4_1) = \sum_{j=0}^{N-1} \prod_{k=1}^j \frac{[n-2+k]}{[k]} [N-k][N+(n-2)+k].$$

For convenience, we introduce the following notations

$$(3.13) \quad f^{SU(n)}(N, k) = \frac{[n-2+k]}{[k]} [N-k][N+(n-2)+k],$$

$$g^{SU(n)}(N, j) = \prod_{k=1}^j f^{SU(n)}(N, k).$$

When $q = \xi_{N-1,a}(s) = \exp(\frac{\pi s \sqrt{-1}}{N+a})$, where $\tilde{a} = a-1$. We have

$$(3.14) \quad [l] = q^l - q^{-l} = 2\sqrt{-1} \sin \frac{ls\pi}{N+a}.$$

It is easy to show that

$$(3.15) \quad f^{SU(n)}(N, k) = \frac{\sin \frac{(n-2+k)s\pi}{N+\tilde{a}}}{\sin \frac{ks\pi}{N+\tilde{a}}} 4 \left(\sin \frac{(k+\tilde{a})s\pi}{N+\tilde{a}} \right) \left(\sin \frac{((n-2)-\tilde{a}+k)s\pi}{N+\tilde{a}} \right).$$

We prove a special case at first. Substituting $\tilde{a} = 0$ (or $a = 1$) to the above formula, we have

$$(3.16) \quad f^{SU(n)}(N, k) = \left(2 \sin \frac{(n-2+k)s\pi}{N} \right)^2.$$

One can show that the function $g^{SU(n)}(N, j) = \prod_{k=1}^j f^{SU(n)}(N, k)$ takes the maximum value at $j = \lfloor (\frac{p}{s} - \frac{1}{6s})N - (n-2) \rfloor$ (here we can take the large N , such that $\frac{N}{s}$ are coprime.) for some $1 \leq p \leq s$.

Therefore, we have

$$(3.17) \quad g^{SU(n)}(N, \lfloor (\frac{p}{s} - \frac{1}{6s})N - (n-2) \rfloor) \leq J_{N-1}^{SU(n)}(4_1) \\ \leq N \cdot g^{SU(N)}(N, \lfloor (\frac{p}{s} - \frac{1}{6s})N - (n-2) \rfloor).$$

Hence

$$(3.18) \quad \frac{\log(g^{SU(n)}(N, \lfloor (\frac{p}{s} - \frac{1}{6s})N - (n-2) \rfloor))}{N} \leq \frac{\log(J_{N-1}^{SU(n)}(4_1))}{N} \\ \leq \frac{\log(N)}{N} + \frac{\log(g^{SU(N)}(N, \lfloor (\frac{p}{s} - \frac{1}{6s})N - (n-2) \rfloor))}{N}.$$

Since $\lim_{N \rightarrow \infty} \frac{\log(N)}{N} = 0$, we have

$$(3.19) \quad \lim_{N \rightarrow \infty} \frac{\log(J_{N-1}^{SU(n)}(4_1))}{N} = \lim_{N \rightarrow \infty} \frac{\log(g^{SU(n)}(N, \lfloor (\frac{p}{s} - \frac{1}{6s})N - (n-2) \rfloor))}{N} \\ = \lim_{N \rightarrow \infty} \frac{2}{N} \sum_{k=1}^{\lfloor (\frac{p}{s} - \frac{1}{6s})N - (n-2) \rfloor} \log(2 | \sin \frac{(n-2+k)s\pi}{N} |)$$

Set $(\frac{p}{s} - \frac{1}{6s})N - (n-2) = M$, then

$$(3.20) \quad \frac{1}{N} = \frac{1}{M+n-2} \left(\frac{p}{s} - \frac{1}{6s} \right),$$

and

$$\begin{aligned}
(3.21) \quad & \frac{2}{N} \sum_{k=1}^{\lfloor (\frac{p}{s} - \frac{1}{6s})N - (n-2) \rfloor} \log(2|\sin \frac{(n-2+k)s\pi}{N}|) \\
&= \frac{2}{M+n-2} (\frac{p}{s} - \frac{1}{6s}) \sum_{k=1}^{\lfloor M \rfloor} \log \left(2 \left| \sin \frac{(n-2+k)(p - \frac{1}{6})\pi}{M + (n-2)} \right| \right) \\
&= \frac{2}{M+n-2} (\frac{p}{s} - \frac{1}{6s}) \sum_{l=n-1}^{\lfloor M \rfloor + n-2} \log \left(2 \left| \sin \frac{l(p - \frac{1}{6})\pi}{M + (n-2)} \right| \right) \\
&= \frac{2}{s\pi} \frac{1}{M+n-2} (p - \frac{1}{6})\pi \sum_{l=1}^{\lfloor M \rfloor + n-2} \log \left(2 \sin \left| \frac{l(p - \frac{1}{6})\pi}{M + (n-2)} \right| \right) \\
&\quad - \frac{2}{M+n-2} (\frac{p}{s} - \frac{1}{6s}) \sum_{l=1}^{n-2} \log \left(2 \left| \sin \frac{l(p - \frac{1}{6})\pi}{M + (n-2)} \right| \right)
\end{aligned}$$

For fixed n, s , when $N \rightarrow \infty$, the second term goes to zero, and the first term is equal to the integral

$$(3.22) \quad \frac{2}{s\pi} \int_0^{(p - \frac{1}{6})\pi} \log(2|\sin t|) dt.$$

Considering the Lobachevsky function

$$(3.23) \quad \Lambda(x) = - \int_0^x \log(2|\sin t|) dt,$$

it has the period π , thus

$$(3.24) \quad \Lambda((p - \frac{1}{6})\pi) = \Lambda(\frac{5\pi}{6}).$$

So we finish the proof for $\tilde{a} = 0$ case.

For $\tilde{a} = n - 2$ ($a = n - 1$), formula (3.15) gives

$$(3.25) \quad f^{SU(n)}(N, k) = \left(2 \sin \frac{(n-2+k)s\pi}{N+n-2} \right)^2.$$

Then we can finish the proof following the similar statement just as $\tilde{a} = 0$ case, or refer to [10] for this case. \square

Proposition 3.6. *For the figure-eight knot 4_1 , part (i) of the Conjecture 3.3 holds; part (ii) of the Conjecture 3.3 holds when $a = 1$ or $a = n - 1$.*

Proof. The part (i) of Conjecture 3.3 is reduced to the congruent relation for 4_1 which has been proved in Theorem 7.11 in [1]. So we only need to prove part (ii) of Conjecture 3.3 when $a = 1$ or $a = n - 1$. By using the property of the Lobachevsky function, we have

$$(3.26) \quad \Lambda(\frac{5\pi}{6}) = -\frac{3}{2}\Lambda(\frac{\pi}{3}).$$

Therefore, by Lemma 3.5

$$\begin{aligned}
 (3.27) \quad 2\pi s \lim_{N \rightarrow \infty} \frac{\log J_{N-1}^{SU(n)}(4_1; \xi_{N-1,a}(s))}{N} &= 4 \int_0^{\frac{5\pi}{6}} \log 2 \sin(x) dx \\
 &= 6\Lambda\left(\frac{\pi}{3}\right) \\
 &= \text{Vol}(S^3 \setminus 4_1).
 \end{aligned}$$

So Proposition 3.6 is proved. \square

In order to prove part (ii) of the Conjecture 3.3 for figure-eight knot 4_1 completely, we first consider the case of $s = 1$.

By the method used in the proof of the Lemma 3.5, the first step is to find a maximal k_m , such that the function of k

$$(3.28) \quad f^{SU(n)}(N, k) = \frac{\sin \frac{(n-2+k)\pi}{N+\tilde{a}}}{\sin \frac{k\pi}{N+\tilde{a}}} 4 \left(\sin \frac{(k+\tilde{a})\pi}{N+\tilde{a}} \right) \left(\sin \frac{((n-2)-\tilde{a}+k)\pi}{N+\tilde{a}} \right),$$

satisfies

$$(3.29) \quad f^{SU(n)}(N, k_m) \geq 1, \quad f^{SU(n)}(N, k_m + 1) < 1.$$

Lemma 3.7. *Such k_m must be in*

$$(3.30) \quad \lfloor \frac{5}{6}(N + \tilde{a}) - 2(n-2) \rfloor \leq k_m \leq \lfloor \frac{5}{6}(N + \tilde{a}) \rfloor.$$

Proof. The upper bound of k_m is clear, in fact, if $k_m \geq \lfloor \frac{5}{6}(N + \tilde{a}) \rfloor$, then $f^{SU(n)}(N, k_m) < 1$. Now we need to estimate a lower bound of k_m . We can assume

$$(3.31) \quad \frac{1}{2} \leq \frac{k_m}{N + \tilde{a}} \leq \frac{5}{6}.$$

Since

$$(3.32) \quad \frac{\sin \frac{(n-2+k)\pi}{N+\tilde{a}}}{\sin \frac{k\pi}{N+\tilde{a}}} = \sin \frac{(n-2)\pi}{N+\tilde{a}} \cot \frac{k\pi}{N+\tilde{a}} + \cos \frac{(n-2)\pi}{N+\tilde{a}},$$

Set $\frac{(n-2)\pi}{N+\tilde{a}} = \alpha$, for $\frac{1}{2} \leq \frac{k}{N+\tilde{a}} \leq \frac{5}{6}$, we have

$$(3.33) \quad \frac{\sin \frac{(n-2+k)\pi}{N+\tilde{a}}}{\sin \frac{k\pi}{N+\tilde{a}}} \geq 1 - \frac{1}{2}\alpha^2 - \sqrt{3}\alpha,$$

where we used the inequalities:

$$(3.34) \quad \sin \alpha < \alpha \text{ and } \cos \alpha > 1 - \frac{1}{2}\alpha^2, \text{ for small } \alpha > 0.$$

Since $\frac{k}{N+\tilde{a}} \geq \frac{1}{2}$, we have

$$(3.35) \quad 4 \sin \frac{(k+\tilde{a})\pi}{N+\tilde{a}} \sin \frac{(n-2-\tilde{a}+k)\pi}{N+\tilde{a}} \geq 4 \left(\sin \frac{(n-2+k)\pi}{N+\tilde{a}} \right)^2 = 4 \left(\sin \left(\frac{5\pi}{6} - \beta \right) \right)^2,$$

where $\beta = \frac{(\frac{5}{6}(N+\tilde{a}) - (n-2) - k)\pi}{N+\tilde{a}}$.

Therefore,

$$(3.36) \quad 4 \sin \frac{(k + \tilde{a})\pi}{N + \tilde{a}} \sin \frac{(n - 2 - \tilde{a} + k)\pi}{N + \tilde{a}} \geq 4(\sin(\frac{5\pi}{6} - \beta))^2 = 1 + 2\sqrt{3} \sin \beta + 2 \sin^2 \beta.$$

Combing (3.33) and (3.36), we have

$$(3.37) \quad \begin{aligned} & \frac{\sin \frac{(n-2+k)\pi}{N+\tilde{a}}}{\sin \frac{k\pi}{N+\tilde{a}}} 4 \left(\sin \frac{(k + \tilde{a})\pi}{N + \tilde{a}} \right) \left(\sin \frac{((n - 2) - \tilde{a} + k)\pi}{N + \tilde{a}} \right) \\ & \geq (1 - \sqrt{3}\alpha - \frac{1}{2}\alpha^2)(1 + 2\sqrt{3} \sin \beta + 2 \sin^2 \beta) \\ & = 1 + \sqrt{3}(2\beta - \alpha) + O(\alpha^2) + O(\beta^2) \end{aligned}$$

when α, β are both small enough.

If we let $k_0 = \frac{5}{6}(N + \tilde{a}) - 2(n - 2)$, then $\beta = \alpha$. By (3.37), we have $f^{SU(n)}(N, k_0) > 1$. Hence, by the definition of k_m , we must have $k_m \geq k_0$. \square

Now, let us consider the case of general s . Without loss of generality, we only need to prove the case that N, s are coprime. For $k = 1$ to $N - 1$, by formula (3.15), The function $f^{SU(n)}(N, k)$ of k has certain period. Thus, as a function of j , one can show that $g^{SU(n)}(N, j)$ may take the maximal value at $j_1 = k_m^{(1)}$ or $j_2 = k_m^{(2)}, \dots, j_p = k_m^{(p)}, \dots$ or $j_s = k_m^{(s)}$, where

$$(3.38) \quad \lfloor \frac{6p-1}{6s}(N + \tilde{a}) - 2(n - 2) \rfloor \leq k_m^{(p)} \leq \lfloor \frac{6p-1}{6s}(N + \tilde{a}) \rfloor.$$

Without loss of generality, we may assume $g^{SU(n)}(N, j)$ take the maximal value at $j_{p_0} = k_m^{(p_0)}$, where $1 \leq p_0 \leq s$. Then $g^{SU(n)}(N, k_m^{(p_0)}) \geq g^{SU(n)}(N, k_m^{(1)}) > 0$ (In fact, $g^{SU(n)}(N, k_m^{(1)}) > 1$ for large N , because the integral $\int_0^{\frac{5\pi}{6}} \log(2 \sin(t)) dt > 0$). Next, we will show the following

Lemma 3.8.

$$(3.39) \quad g^{SU(n)}(N, k_m^{p_0}) \leq J_{N-1}^{SU(n)}(4_1) \leq N \cdot g^{SU(n)}(N, k_m^{p_0}).$$

Proof. By definition

$$(3.40) \quad J_{N-1}^{SU(n)}(4_1) = 1 + \sum_{j=1}^{N-1} g^{SU(n)}(N, j) \leq N \cdot g^{SU(n)}(N, k_m^{p_0})$$

on the other hand side, we let

$$(3.41) \quad \begin{aligned} k^{(1)} &= \lfloor \frac{(N + \tilde{a})}{s} \rfloor - (n - 2), \\ k^{(2)} &= \lfloor \frac{2(N + \tilde{a})}{s} \rfloor - (n - 2), \\ &\dots \\ k^{(p_0-1)} &= \lfloor (p_0 - 1) \frac{(N + \tilde{a})}{s} \rfloor - (n - 2). \end{aligned}$$

In the summation of $\sum_{j=1}^{N-1} g^{SU(n)}(N, j)$, the possible negative terms are only in the following list:

$$(3.42) \quad g^{SU(n)}(N, k^{(i)} + l), \text{ for } i = 1, \dots, p_0 - 1, \text{ and } l = 0, \dots, n - 2.$$

Since $k^{(i)} + l = \lfloor \frac{i(N+\tilde{a})}{s} \rfloor - (n-2) + l = \frac{i(N+\tilde{a})}{s} - \langle \frac{i(N+\tilde{a})}{s} \rangle - (n-2) + l$, where we use $\langle \frac{i(N+\tilde{a})}{s} \rangle$ to denote the fractional part of $\frac{i(N+\tilde{a})}{s}$, hence $0 \leq \langle \frac{i(N+\tilde{a})}{s} \rangle < 1$. Therefore

$$(3.43) \quad \begin{aligned} \sin \frac{(x + k^{(i)} + l)s\pi}{N + \tilde{a}} &= \sin \left(i\pi - \frac{(\langle \frac{i(N+\tilde{a})}{s} \rangle + (n-2) - l - x)s\pi}{N + \tilde{a}} \right) \\ &= (-1)^{i-1} \sin \left(\frac{(\langle \frac{i(N+\tilde{a})}{s} \rangle + (n-2) - l - x)s\pi}{N + \tilde{a}} \right), \end{aligned}$$

where x stands for $0, n-2, n-2-\tilde{a}$ and \tilde{a} . So for large N , $\sin \frac{(x+k^{(i)}+l)s\pi}{N+\tilde{a}} = O(\frac{1}{N+\tilde{a}})$. Thus, we have

$$(3.44) \quad f^{SU(n)}(N, k^{(i)} + l) = O\left(\left(\frac{1}{N + \tilde{a}}\right)^2\right),$$

then

$$(3.45) \quad \prod_{j=0}^l f^{SU(n)}(N, k^{(i)} + j) = O\left(\left(\frac{1}{N + \tilde{a}}\right)^{2(l+1)}\right)$$

and

$$(3.46) \quad 1 + \sum_{l=0}^{n-2} \prod_{j=0}^l f^{SU(n)}(N, k^{(i)} + j) > 0.$$

Therefore,

$$(3.47) \quad g^{SU(n)}(N, k^{(i)} - 1) + \sum_{l=0}^{n-2} g^{SU(n)}(N, k^{(i)} + l) = g^{SU(n)}(N, k^{(i)} - 1) \left(1 + \sum_{l=0}^{n-2} \prod_{j=0}^l f^{SU(n)}(N, k^{(i)} + j) \right) > 0$$

Thus we have

$$\begin{aligned} J_N^{SU(n)}(4_1) &= \sum_{\substack{u = 0 \text{ and } u \neq k^{(i)} + l, \\ \text{where } l = -1, 0, \dots, n-2}} g^{SU(n)}(N, u) \\ &\quad + \sum_{i=0}^s \left(g^{SU(n)}(N, k^{(i)} - 1) + \sum_{l=0}^s g^{SU(n)}(N, k^{(i)} + l) \right) \\ &> \sum_{\substack{u = 0 \text{ and } u \neq k^{(i)} + l, \\ \text{where } l = -1, 0, \dots, n-2}}^N g^{SU(n)}(N, u) \end{aligned}$$

It is easy to know that each term $g^{SU(n)}(N, u)$ in the above expression is positive. For large N , it is impossible that $k^{(i)} + l = k_m^{(j)}$ for any pair (i, j, l) , where $1 \leq i, j \leq s, l = -1, 0, \dots, n-2$. Thus we have

$$(3.48) \quad J_N^{SU(n)}(4_1) = \sum_{u=0}^N g^{SU(n)}(N, u) \geq g^{SU(n)}(N, k_m^{(p_0)})$$

□

Proposition 3.9. *Part (ii) of the Conjecture 3.3 holds for the figure-eight knot 4_1 .*

Proof. Now, we can finish the proof of proposition as follow:

$$(3.49) \quad \begin{aligned} & \lim_{N \rightarrow \infty} \frac{\log(J_{N-1}^{SU(n)}(4_1))}{N} \\ &= \lim_{N \rightarrow \infty} \frac{\log(g^{SU(n)}(N, k_m^{(p_0)}))}{N} \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{k_m^{(p_0)}} \log \left(4 \frac{|\sin \frac{(n-2+k)s\pi}{N+\tilde{a}}|}{|\sin \frac{ks\pi}{N+\tilde{a}}|} \left| \sin \frac{(k+\tilde{a})s\pi}{N+\tilde{a}} \right| \left| \sin \frac{((n-2)-\tilde{a}+k)s\pi}{N+\tilde{a}} \right| \right) \end{aligned}$$

where we have used the fact $g^{SU(n)}(N, k_m^{(p_0)}) > 0$, thus the number of the negative term of the form $\sin \frac{(x+k)s\pi}{N+\tilde{a}}$ in the product $g^{SU(n)}(N, k_m^{(p_0)})$ must be even. For each term of above with the form $\log \left(\left| \sin \frac{(x+k)s\pi}{N+\tilde{a}} \right| \right)$, by the method in the proof of Lemma 3.5, we have

$$(3.50) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{k_m^{(p_0)}} \log \left(\left| \sin \frac{(x+k)s\pi}{N+\tilde{a}} \right| \right) = \frac{1}{\pi} \int_0^{\frac{5}{6}\pi} \log |\sin(t)| dt.$$

Finally, we obtain

$$(3.51) \quad \lim_{N \rightarrow \infty} \frac{\log(J_{N-1}^{SU(n)}(4_1))}{N} = \left(\frac{p_0}{s} - \frac{1}{6s} \right) \log 4 + \frac{2}{s\pi} \int_0^{(p_0 - \frac{1}{6})\pi} \log |\sin(t)| dt = \frac{2}{s\pi} \int_0^{(p_0 - \frac{1}{6})\pi} \log(2|\sin(t)|) dt.$$

Finally we have

$$(3.52) \quad 2\pi s \lim_{N \rightarrow \infty} \frac{\log J_{N-1}^{SU(n)}(4_1; \xi_{N-1,a}(s))}{N} = 4 \int_0^{(p_0 - \frac{1}{6})\pi} \log(2|\sin(t)|) dt = -4\Lambda\left((p_0 - \frac{1}{6})\pi\right)$$

$$(3.53) \quad = -4\Lambda\left(\frac{5}{6}\pi\right) = 6\Lambda\left(\frac{\pi}{3}\right) = \text{Vol}(S^3 \setminus 4_1).$$

□

4. APPENDIX

4.1. **Example of Conjectures 1.4.** We define

$$(4.1) \quad Q(N, n, a, s) = 2\pi s [\log J_N^{SU(n)}(\mathcal{K}; \xi_{N,a}(s)) - \log J_{N-1}^{SU(n)}(\mathcal{K}; \xi_{N-1,a}(s))],$$

and compute its limit $\lim_{N \rightarrow \infty} Q(N, n, a, s)$.

4.1.1. *The knot 5_2 .* We know that $Vol(S^3/5_2) \approx 2.828122$ and $0.1532041333 * 2\pi^2 = 3.02413$. So the complex volume for hyperbolic knot 5_2 is $2.828122 + 3.02413\sqrt{-1}$. By using the formula for $SU(n)$ invariant of 5_2 in [11]. We have the following table. Here we set $s = 1$.

$N \setminus (n, a)$	(2, 1)	(3, 1)	(3, 2)
10	$3.73795 + 2.62595\sqrt{-1}$	$4.95561 + 1.83803\sqrt{-1}$	$4.77077 + 2.02852\sqrt{-1}$
20	$3.27786 + 2.92530\sqrt{-1}$	$3.90996 + 2.71482\sqrt{-1}$	$3.85936 + 2.74299\sqrt{-1}$
30	$3.13249 + 2.97960\sqrt{-1}$	$3.55378 + 2.88496\sqrt{-1}$	$3.53064 + 2.89368\sqrt{-1}$
40	$3.05822 + 2.99885\sqrt{-1}$	$3.37391 + 2.94535\sqrt{-1}$	$3.36071 + 2.94911\sqrt{-1}$
50	$3.01308 + 3.00786\sqrt{-1}$	$3.26546 + 2.97352\sqrt{-1}$	$3.25694 + 2.97547\sqrt{-1}$
70	$2.96096 + 3.01577\sqrt{-1}$	$3.14105 + 2.99819\sqrt{-1}$	$3.13666 + 2.99891\sqrt{-1}$
100	$2.92148 + 3.02001\sqrt{-1}$	$3.04745 + 3.01138\sqrt{-1}$	$3.04528 + 3.01163\sqrt{-1}$
200	$2.87502 + 3.02309\sqrt{-1}$	$2.93794 + 3.02093\sqrt{-1}$	$2.93739 + 3.02096\sqrt{-1}$

$N \setminus (n, a)$	(4, 1)	(4, 2)	(4, 3)
10	$6.23105 + 0.579569\sqrt{-1}$	$5.80661 + 0.952116\sqrt{-1}$	$5.70074 + 1.31854\sqrt{-1}$
20	$4.56915 + 2.40633\sqrt{-1}$	$4.43367 + 2.42953\sqrt{-1}$	$4.41159 + 2.51307\sqrt{-1}$
30	$3.98932 + 2.74861\sqrt{-1}$	$3.92527 + 2.74711\sqrt{-1}$	$3.91657 + 2.78186\sqrt{-1}$
40	$3.69821 + 2.86884\sqrt{-1}$	$3.66116 + 2.86452\sqrt{-1}$	$3.65662 + 2.88323\sqrt{-1}$
50	$3.52356 + 2.92461\sqrt{-1}$	$3.49946 + 2.92049\sqrt{-1}$	$3.49671 + 2.93210\sqrt{-1}$
70	$3.32419 + 2.97327\sqrt{-1}$	$3.31168 + 2.97036\sqrt{-1}$	$3.31036 + 2.97605\sqrt{-1}$
100	$3.17494 + 2.99918\sqrt{-1}$	$3.16873 + 2.99744\sqrt{-1}$	$3.16812 + 3.00014\sqrt{-1}$
200	$3.00124 + 3.01788\sqrt{-1}$	$2.99967 + 3.01736\sqrt{-1}$	$2.99953 + 3.01800\sqrt{-1}$

From the above tables, one can see that the $SU(2)$ invariants, i.e., the colored Jones polynomials, converge to the complex volume faster than the general $SU(n)$ invariants. As to the $SU(n)$ invariants, one can see that the $SU(n)$ invariants converge to the complex volume at $a = n - 1$ faster than at the other values $a = 1, \dots, n - 2$.

4.2. **Examples of Conjecture 1.3.** Case of knot 5_2

The $SU(2)$ invariant:

$$H_0 = 1$$

$$H_1 = -q^4(1 + q^4)$$

$$H_2 = q^{10}(1 + q^4 + q^6 + q^{12})$$

$$H_3 = -q^{18}(1 + q^4 + q^6 + q^8 + q^{12} + q^{14} + q^{16} + q^{24})$$

$$H_4 = q^{28}(1 + q^4 + q^6 + q^8 + q^{10} + q^{12} + q^{14} + 2q^{16} + q^{18} + q^{20} + q^{24} + q^{26} + q^{28} + q^{30} + q^{40})$$

The $SU(3)$ invariant:

$$H_0 = 1$$

$$H_1 = -q^5(1 + q^2)^2(1 - q^2 + q^4)$$

$$H_2 = q^{12}(1 + q^2 + q^4)(1 + q^6 + q^8 + q^{16})$$

$$\begin{aligned}
H_3 &= -q^{21}(1+q^2)^2(1+q^4)(1-q^2+q^4+q^8+q^{16}+q^{20}-q^{22}+q^{24}-q^{26}+q^{28}) \\
H_4 &= q^{32}(1+q^2+q^4+q^6+q^8)(1+q^6+q^8+q^{10}+q^{12}+q^{16}+q^{18}+2q^{20}+q^{22}+q^{24}+ \\
&\quad q^{30}+q^{32}+q^{34}+q^{36}+q^{48})
\end{aligned}$$

The $SU(4)$ invariant:

$$\begin{aligned}
H_0 &= 1 \\
H_1 &= -q^6(1+q^2+q^4+q^8+q^{10}+q^{12}) \\
H_2 &= q^{14}(1+q^2+q^4)(1+q^4+q^8+q^{10}+q^{12}+q^{14}+q^{20}+q^{24}) \\
H_3 &= -q^{24}(1+q^4)^2(1+q^2+2q^8+2q^{10}+q^{12}+q^{14}+2q^{16}+q^{18}+q^{22}+3q^{24}+2q^{26}+q^{32}+q^{38}+q^{40}) \\
H_4 &= q^{36}(1+q^2+q^4)(1+q^2+q^4+q^6+q^8)(1-q^2+q^4+q^8+q^{12}+q^{14}+q^{18}+q^{20}+ \\
&\quad 2q^{24}+2q^{28}+q^{32}+q^{36}+q^{40}+q^{42}+q^{46}+q^{56}-q^{58}+q^{60})
\end{aligned}$$

Case of knot 6_1

The $SU(2)$ invariant:

$$\begin{aligned}
H_0 &= 1 \\
H_1 &= 1+q^4 \\
H_2 &= 1+q^4+q^6+q^{12} \\
H_3 &= 1+q^4+q^6+q^8+q^{12}+q^{14}+q^{16}+q^{24} \\
H_4 &= 1+q^4+q^6+q^8+q^{10}+q^{12}+q^{14}+2q^{16}+q^{18}+q^{20}+q^{24}+q^{26}+q^{28}+q^{30}+q^{40}
\end{aligned}$$

The $SU(3)$ invariant:

$$\begin{aligned}
H_0 &= 1 \\
H_1 &= q^{-1}(1+q^2)^2(1-q^2+q^4) \\
H_2 &= q^{-2}(1+q^2+q^4)(1+q^6+q^8+q^{16}) \\
H_3 &= q^{-3}(1+q^2)^2(1+q^4)(1-q^2+q^4+q^8+q^{16}+q^{20}-q^{22}+q^{24}-q^{26}+q^{28}) \\
H_4 &= q^{-4}(1+q^2+q^4+q^6+q^8)(1+q^6+q^8+q^{10}+q^{12}+q^{16}+q^{18}+2q^{20}+q^{22}+q^{24}+ \\
&\quad q^{30}+q^{32}+q^{34}+q^{36}+q^{48})
\end{aligned}$$

The $SU(4)$ invariant:

$$\begin{aligned}
H_0 &= 1 \\
H_1 &= q^{-2}(1+q^2+q^4+q^8+q^{10}+q^{12}) \\
H_2 &= q^{-4}(1+q^2+q^4)(1+q^4+q^8+q^{10}+q^{12}+q^{14}+q^{20}+q^{24}) \\
H_3 &= q^{-6}(1+q^4)^2(1+q^2+2q^8+2q^{10}+q^{12}+q^{14}+2q^{16}+q^{18}+q^{22}+3q^{24}+2q^{26}+q^{32}+q^{38}+q^{40}) \\
H_4 &= q^{-8}(1+q^2+q^4)(1+q^2+q^4+q^6+q^8)(1-q^2+q^4+q^8+q^{12}+q^{14}+q^{18}+q^{20}+ \\
&\quad 2q^{24}+2q^{28}+q^{32}+q^{36}+q^{40}+q^{42}+q^{46}+q^{56}-q^{58}+q^{60})
\end{aligned}$$

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